assumption in the absence of any theory to the contrary. The scatter of his data is evident from Figure 4.

The resistance curves of Figure 3 can be used to design pipe lines for a wide spectrum of non-Newtonian slurries. They represent the industrially important range of non-Newtonian behavior. Further investigation will be required in order to develop a mechanistic explanation for the results of Figure 2 and to verify experimentally the maximum in the high N_{He} curves.

ACKNOWLEDGMENT

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NOTATION

B = damping parameter in turbulent momentum flux model
D = pipe diameter
dp/dz = axial pressure gradient
dv_z/dr = shear rate
E = exponential factor as defined by Equation (12)
f = Fanning friction factor
f_c = critical laminar-turbulent transition value of f
G = shear function as defined by Equation (14)
k = mixing length constant, $k = 0.36$
L = modified mixing length as defined by Equation (8)
N_{He} = Hedstrom number, $D^2\rho\tau_0/\eta^2$
N_{Re} = Reynolds number, $D\langle v_z\rangle\rho/\eta$
N_{Rec} = critical laminar-turbulent value of N_{Re}
q = $\langle v_z\rangle/R_w$, pseudoshear rate
r = radial position variable
R = $N_{Re}\sqrt{f}$
R_c = $N_{Rec}\sqrt{f_c}$
R_w = pipe wall radius
u^+ = v_z/v^*
v_z = axial velocity component
$\langle v_z\rangle$ = area mean value of v_z
v^* = $\sqrt{\tau_w/\rho}$

Greek Letters

η = plastic viscosity in Equation (1)
ξ = $r/R_w = \tau_{rz}/\tau_w$
ξ_0 = τ_0/τ_w
ξ_{0c} = τ_0/τ_{wc}
ρ = fluid density
τ_{rz} = axial shear stress or momentum flux
τ_w = value of τ_{rz} at $r = R_w$
τ_{wc} = value of τ_w when $N_{Re} = N_{Rec}$
τ_0 = yield stress in Equation (1)
ϕ = mixing length parameter as defined by Equation (9)

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Waves on a Thin Film of Viscous Liquid

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In this paper we consider the problem of the flow of a viscous incompressible fluid down an inclined wall. A solution is obtained by assuming that the free surface is a wave of low frequency. The solution is numerical and the results are compared with existing theories and available experimental results.

In this paper we will consider the flow of a viscous liquid in a thin film. Such flows are often observed in everyday life; for example, when rain runs down a window pane or when paint drains from some solid object which has been dipped in it. This is also a subject of importance in chemical engineering and has been studied by experimenters in that field. The character of the flow has been shown to depend largely on the Reynolds number, although surface tension is important in most cases. For example, experiments show that in flow down a vertical wall, the motion is turbulent when N_{Re} is greater than 300 (2). When N_{Re} is less, the mean flow is governed by a law of laminar friction, and the mean depth is approximately that given by the simple theory of Nusselt (8) and Jeffreys (2), who assume uniform Poiseuille flow. Nevertheless, waves of various amplitudes are observed in all laminar flows, except those at very small N_{Re}.

The stability of the uniform solution given by Nusselt and Jeffreys has been studied by several authors (5, 10, 1, 9). These theoreticians have shown that there is a critical value of N_{Re} that depends on the value of the surface tension and the angle of the inclination of the wall, such that only flows that have N_{Re} less than this critical value are stable. When the wall is vertical, Benjamin has shown that the critical Reynolds number is zero; that is, all flows are unstable. However, he also shows that for small Reynolds number the amplification of the most unstable wave is very small, but that it increases rapidly as N_{Re} increases above 4. This agrees well with experiment.

Thus the onset of instability of the uniform Poiseuille flow is well understood. The problem of finding the free surface profile after the onset of instability has not received much attention. In fact, the various authors that have turned their attention to the problem have studied only infinitesimal disturbances of the uniform solution. This theory gives undamped sine waves. There are many observations that are not explained by this theory and there are many reasons to doubt the validity ot this theory. For example, in most observed waves the solution is not a symmetric curve like a sine wave, even when the amplitude is small. Also in several cases, observed waves can have amplitudes comparable with the mean depth of the water.

Here we will use a nonlinear theory rather similar to the cnoidal wave theory of inviscid flow, in the hope that the results of the theory will apply to finite amplitude waves.

THE SIGNIFICANT RESULTS FROM PRIOR WORK

The first major contribution was the solution given by Nusselt (8). As mentioned before, he assumed the flow to be uniform and obtained Poiseuille flow. His results take on the form

$$u = -\frac{3u_0}{h_0}\left(y - \frac{y^2}{2h_0}\right) \tag{1}$$

where

$$u_0 = -\frac{1}{h_0}\int_0^{h_0} u\, dy = \frac{1}{3}\,\frac{gh_0^2}{\nu}\sin\theta \tag{2}$$

This theory is invalid when waves start to appear, but is widely used as a first approximation. Later investigators (4, 6, 7, 9, 11) have, in fact, started from the assumption (1) in a local sense where u_0 and h_0 vary slowly; that is, they assume that

$$u = -3\,\frac{u_0(x,t)}{h_0(x,t)}\left\{ y - \frac{1}{2}\, y^2/h_0(x,t)\right\} \tag{3}$$

They then use the x momentum equation to find an equation for the height. To eliminate the pressure, they also have to assume that the pressure gradient perpendicular to the wall is the same as that given by the hydrostatic approximation. The height is linearized with respect to the departure η from the mean height, and a third-order differential equation for η is obtained. We shall not assume that the solution has the form (3) nor that the pressure across the film is hydrostatic. This gives rise to extra terms in the final equation for η which are of the same order as those already obtained. However, we shall assume that the wavelength of the free surface disturbance is large. This assumption will be explained fully later. We will also keep the significant nonlinear terms and discuss how they affect the linear solution.

THE BASIC EQUATIONS

The motion is assumed to be two-dimensional flow of an incompressible viscous fluid down a wall of inclination θ. The x axis is taken in this direction and the y axis perpendicular to it. This system of coordinate axes is shown in Figure 1.

The equations governing the motion are the equation of continuity and the Navier-Stokes momentum equations. These are to be solved with boundary conditions

$$u = v = 0 \quad \text{on} \quad y = h \tag{4}$$

and

$$\frac{\partial h}{\partial t} + \frac{u\partial h}{\partial x} - v = 0 \quad \text{on} \quad y = h \tag{5}$$

Also, we obtain two equations from the continuity of stress across the free surface,

$$-\frac{1}{\rho}\,\mathrm{p}\cdot n = \left(\frac{\Gamma}{r} - \frac{p_a}{\rho}\right) n \tag{6}$$

The pressure distribution is found by integrating the y component of the Navier-Stokes equations and using boundary condition (6). The resulting equation is then simplified by the introduction of a stream function, defined by $u = \partial\psi/\partial y$, $v = -\partial\psi/\partial x$, so that the continuity equation is satisfied automatically. Since only solutions that are periodic in x and t are considered, the problem can be simplified by assuming that the dependence on x and t can be expressed in terms of the single variable $\chi = x + ct$.

When h is constant we obtain the uniform flow solution

$$\psi_{00} = \frac{g\sin\theta}{\nu}\, h^3\left[\frac{1}{3!}\left(\frac{y}{h}\right)^3 - \frac{1}{2!}\left(\frac{y}{h}\right)^2\right] \tag{7}$$

We now use this result to introduce nondimensional variables by choosing a typical height h_0 as that attained by uniform Poiseuille flow with a given flow rate Q. This also introduces a Reynolds number N_{Re}. Therefore

$$-\psi_{00}(h_0) = Q = \nu N_{Re} \tag{8}$$

and

$$h_0 = \{3\,\nu\, Q/(g\sin\theta)\}^{1/3} \tag{9}$$

This gives a typical velocity scale

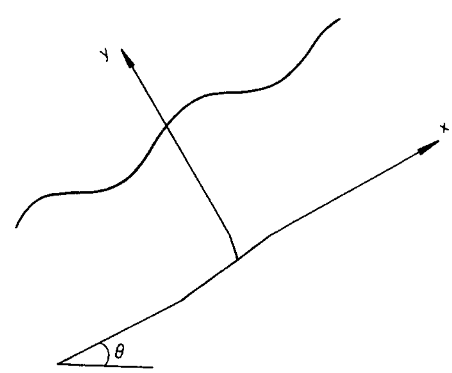

Fig. 1. The coordinate axes.

$$u_0 = -\frac{1}{h_0}\int_0^{h_0} \frac{\partial \psi_{00}}{\partial y}\, dy = \frac{1}{3}\, g \sin\theta\, h_0^2/\nu \quad (10)$$

and hence the Froude number as

$$N_{Fr} = u_0/\sqrt{g h_0} \quad (11)$$

We also introduce a nondimensional wave speed as

$$\alpha = c/u_0 \quad (12)$$

In introducing the nondimensional variables we stretch the y coordinates with respect to the x coordinate. This results from anticipating that the free surface disturbance is a wave of low frequency for which the x derivatives are in decreasing order of magnitude. This assumes that the wave length is very much greater than the height, an assumption that Figure 5 shows to be satisfied experimentally. We also introduce a nondimensional parameter a, which is the amplitude of the free surface motion. Thus the nondimensional variables are introduced as

$$Y = y/h_0,\quad X = \epsilon\chi/h_0,\quad \Psi = \psi/(u_0 h_0),\quad h = h_0(1 + a\eta) \quad (13)$$

The equation for Ψ then becomes

$$\epsilon[\Psi_Y + \alpha]\Psi_{YX} - \epsilon\Psi_{YY}\Psi_X$$
$$= \frac{\epsilon^3 a\Gamma}{u_0^2 h_0}\frac{\partial}{\partial X}[\eta_{XX}(1 + a^2\epsilon^2\eta_X^2)^{-3/2}] - \frac{\epsilon a}{N_{Fr}^2}\cos\theta\,\eta_X$$
$$+ \epsilon\frac{\partial}{\partial X}\int_Y^{1+a\eta}\left\{\epsilon^2[(\Psi_Y + \alpha)\Psi_{XX} - \Psi_{XY}\Psi_X] - \frac{\sin\theta}{3N_{Fr}^2}\epsilon[\Psi_{XYY} + \epsilon^2\Psi_{XXX}]\right\}dY$$
$$+ \frac{2\sin\theta\,\epsilon^2}{3N_{Fr}^2}\frac{\partial}{\partial X}[\Psi_{XY}(1 + a^2\epsilon^2\eta_X^2)/(1 - a^2\epsilon^2\eta_X^2)]_{Y=h}$$
$$+ \frac{\sin\theta}{3N_{Fr}^2}[\Psi_{YYY} + \epsilon^2\Psi_{YXX}] - \frac{\sin\theta}{N_{Fr}^2} \quad (14)$$

To solve this equation we assume that a and ϵ are small and expand the stream function in powers of a and ϵ as

$$\Psi = \Psi^{00}(Y) + \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} a^i\epsilon^j\,\Psi^{ij}(X, Y) \quad (15)$$

The boundary condition expressing continuity of tangential stress at the free surface is expanded to give a condition at $Y = 1$ in the following way:

$$G(X, 1 + a\eta) = G(X, 1) + \sum_{i=1}^{\infty}\frac{(a\eta)^i}{i!}\left(\frac{\partial^i G}{\partial Y^i}\right)_{Y=1} \quad (16)$$

This method of expansion is similar but more direct than the one proposed by Levich and Krylov (7), who assume that Ψ can be expanded as a polynomial in Y whose coefficients are functions of χ only. [Ψ^{1n} introduced in Equation (15) turns out to be a polynomial in Y whose coefficients are proportional to $d^n\eta/d\chi^n$.] They truncate the series after N terms and introduce the expansion into Equation (14). A set of nonlinear simultaneous ordinary differential equations can then be obtained by multiplying by Y^i and integrating with respect to Y from 0 to $1 + \alpha\eta$. These are then solved with the existing boundary conditions [equations corresponding to (4), (5), and (6)].

By equating coefficients of $a^i\epsilon^j$ to zero in Equation (14), we obtain an equation for $\Psi^{ij}(X,Y)$ and by equating coefficients of $a^i\epsilon^j$ to zero in Equations (4) and (6), we obtain three boundary conditions for Ψ^{ij}. Thus formally we can obtain the solution for all the Ψ^{ij}. The final equation for η then comes from the integrated form of Equation (5)

$$\Psi + \alpha Y = \text{constant},\quad \text{on}\quad Y = 1 + a\eta \quad (17)$$

or

$$\Psi(X, 1) + \Sigma\frac{(a\eta)^i}{i!}\left[\frac{\partial^i\Psi(X, Y)}{\partial Y^i}\right]_{Y=1} + \alpha(1 + a\eta) = \text{const.} \quad (18)$$

We will approximate by taking terms up to order $a\epsilon^2$ and $a^2\epsilon$. This at first sight seems to assume that the amplitude and wave number are both small. However, we will see from our solutions that our approximation requires ϵN^2_{Fr} be small rather than just ϵ. Also, as we wish to include the effects of surface tension we will take the terms of order $a\epsilon^3$ that are multiplied by $\Gamma/(u_0^2 h_0)$. This assumes that the surface tension ratio is large. This means that the inverse ratio or Weber number defined by

$$N_{We} = u_0^2 h_0/\Gamma \quad (19)$$

is small.

The systematic solution of Equation (14) then gives

$$\Psi^{00}(X, Y) = 3(Y^3/6 - Y^2/2), \quad (20)$$

$$\Psi^{10}(X, Y) = -3\eta y^2/2 \quad (21)$$

$$\Psi^{11}(X, Y) = 3\eta'\cot\theta(Y^3/3 - Y^2/2) - 9N_{Fr}^2\operatorname{cosec}\theta\eta'[\alpha(Y^4/4! - Y^2/4) - 3(Y^5/5! - Y^2/12)] \quad (22)$$

$$\Psi^{12}(X, 1) = -\eta''\{3 + 27N_{Fr}^2\operatorname{cosec}^2\theta[\cos\theta(277/7! - 2\alpha/45) + N_{Fr}^2(3879 - 7221\alpha + 3461\alpha^2)/8!]\} \quad (23)$$

$$\Psi^{20}(X, 1) = 0 \quad (24)$$

$$\Psi^{21}(X, 1) = -\eta\eta'(N_{Fr}^2\operatorname{cosec}\theta(351/40 - 9\alpha/2) + 3/2\cot\theta) \quad (25)$$

$$\Psi^{13}(X, 1) = N_{Fr}^2/N_{We}\operatorname{cosec}\theta\eta''' + 0(1) \quad (26)$$

The final equation for η is then obtained from (18) as

$$A\eta''' + B\eta'' + (C\eta + D)\eta' + E\eta + F\eta^2 = 0 \quad (27)$$

where A, B, C, D, E, and F are constants.

THE EXISTENCE OF PERIODIC SOLUTIONS

The final equation for the departure η from the basic height is a third-order nonlinear differential equation with constant coefficients. If the equation is linearized about $\eta = 0$, the equation becomes

$$A\eta''' + B\eta'' + D\eta' + E\eta = 0 \quad (28)$$

This equation has as solutions

$$\eta = \sum_{i=1}^{3} A_i\, e^{\beta_i X} \quad (29)$$

where A_i are constants and the β_i are roots of the cubic equations

$$A\beta^3 + B\beta^2 + D\beta + E = 0 \quad (30)$$

The only periodic solution that is bounded occurs when two of the roots of Equation (30) are pure imaginary. Then the solution is just a sine wave. We also require that all solutions of the form (29) tend to a sine wave as $X \to -\infty$, so we must have the real root greater than

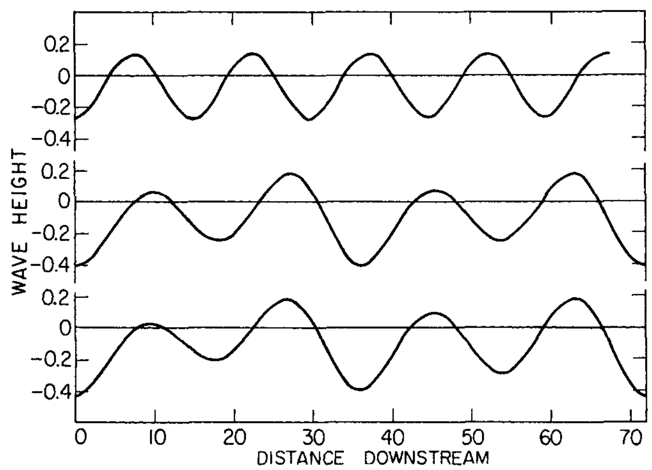

Fig. 2. Typical wave profiles for different wave velocities; Reynolds number in the region of 10.

zero. Massot, Irani, and Lightfoot (7) obtained an equation similar to Equation (27) and proposed that this was the only bounded solution. They gave a relation between certain physical parameters so that the roots of the equation corresponding to (30) had two pure imaginary roots, that is, $AE = BD$. However, these solutions have an arbitrary amplitude and no acceptable method for determining the amplitude was found. Moreover these solutions are only valid when the amplitude is small. If, however, the real parts of the two complex roots of Equation (30) are negative, then the oscillating part of the linear solution will have a growing amplitude as the waves proceed downstream. This is analogous to inviscid irrotational waves where the linear sine wave is only a good approximation to the nonlinear cnoidal wave when the amplitude is small. Also the nonlinear cnoidal wave provides a definite amplitude from the initial conditions. Again when the stream is supercritical, the linear solutions are exponentials which are modified by the nonlinear terms to form waves. So we hope that the nonlinear terms in Equation (27) will modify the linear solution of (28) in such a manner as to give a finite amplitude bounded solution with a given amplitude.

DISCUSSION OF THE NUMERICAL RESULTS

Equation (27) was integrated for the case of water flowing down a vertical wall. Since we are interested in solutions that tend to a periodic solution as we tend downstream, the initial boundary conditions at $X = 0$ become unimportant. In fact, when a periodic solution of Equation (27) was found, all solutions tended to this periodic solution as $X \rightarrow -\infty$. We first notice that the coefficient of η'' in Equation (27) is negative. This means that when the parameters N_{Fr} and α are chosen so that a solution of the linear Equation (28) is an undamped sine wave, all solutions tend to this solution as $X \rightarrow -\infty$.

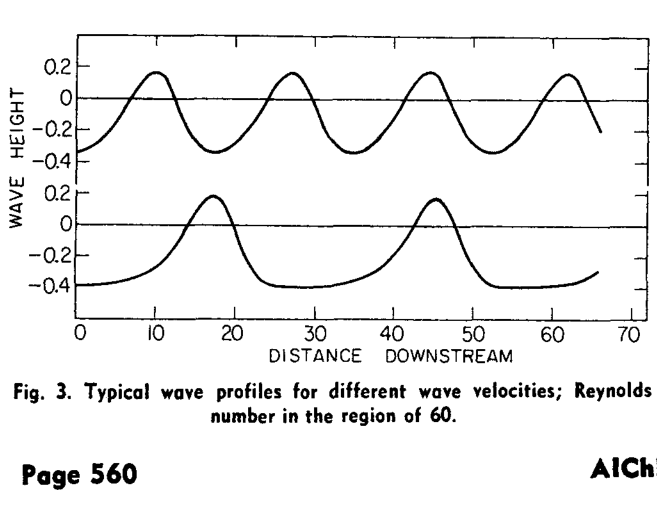

Fig. 3. Typical wave profiles for different wave velocities; Reynolds number in the region of 60.

Wave Description

The parameters N_{Fr} and α were first chosen so that the corresponding cubic equation (30) had two complex roots whose real part was zero. When the differential equation was integrated with arbitrary initial conditions, it was found that all waves were damped out as the integration advanced downstream. This indicated that the undamped sine wave of the linear equation (28) was indeed only a solution for infinitesimal amplitudes.

The parameters were then altered so that the real part of the complex roots of the equation corresponding to (30) was negative. Then when the resulting equation was integrated, it was found that all solutions tended to the same solution as $X \rightarrow -\infty$. When the amplitude was small, the solution was a single-peaked asymmetric oscillation which was steeper on the downstream side of the crest than on the upstream.

As the parameters were altered so that the real part of the complex roots of the equation corresponding to (30) became more negative, it was found that in general the solution fell into two categories. For high values of the Weber number, that is, thick films, the solution remained a single-peaked asymmetric oscillation whose amplitude and wavelength increased until finally it was impossible to find a bounded solution. For low values of the Weber number, that is, thin films, the single-peaked solution split up into a double-peaked solution. Then the double oscillation split up into a four-peaked oscillation. Next the four-peaked solution split up into a very irregular solution in which no periodicity could be found. Finally, as in the case where only a single-peaked oscillation existed, no bounded solution was found. Thus for given flow conditions we have a number of possible steady profiles. Examples of these solutions are shown in Figures 2 and 3.

Wavelength and Wave Velocity

Figure 4 gives the nondimensional wave velocity of infinitesimal sine waves on water. The curve, however, varies from fluid to fluid, not as in the theory of Massot, Irani, and Lightfoot (7). The present theory shows that when θ, N_{fp}, and N_{Fr} are given, there exists a range of wave velocities less than the critical wave velocity for infinitesimal sine waves, for which a steady periodic solution to Equation (27) exists. For each wave velocity the wavelength and amplitude are unique. In fact, for given

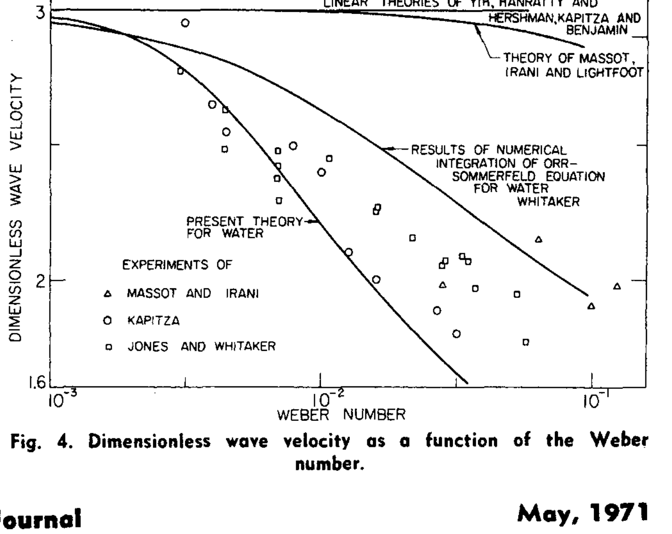

Fig. 4. Dimensionless wave velocity as a function of the Weber number.

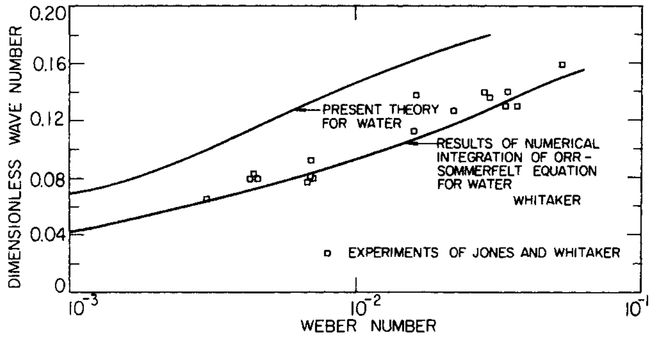

Fig. 5. Dimensionless wave number as a function of the Weber number.

flow conditions it was found that the wavelength and amplitude both increased as the wave velocity decreased from the critical wave velocity.

The wave number ($2\pi h_0$/wavelength) is given as a function of the Weber number in Figure 5. Also plotted are the experimental results of Massot and Irani, Jones and Whitaker, and Kapitza. These experimental results indicate clearly that the assumption of small wave number, that is, small ϵ, is valid even for a Weber number of 10^{-1}, which is well outside the range of Weber numbers where surface tension is dominant.

However, Figures 4 and 5 also show the results of the numerical integrations of the Orr-Sommerfeld equation. These results were obtained by Whitaker (9) and represent the wave velocity of the most unstable infinitesimal wave. We would expect that the present theory for the case of zero amplitude and the theory of Whitaker would be in reasonable agreement. The two theories differ most when the Weber number, and hence the Reynolds number or Froude number, is large. In this case we can show that the neglected terms that are linear in a (that is, the terms $a\epsilon^j$ for $j \geqq 3$) have a large effect. Equation (27) shows that the coefficient of $a\epsilon$ has a term proportional to N_{Fr}^2 and the coefficient of $a\epsilon^2$ has a term proportional to N_{Fr}^4. It is easy to deduce that the coefficient of $a\epsilon^j$ has a term proportional to N_{Fr}^{2j}. Thus the initial assumption of small ϵ restricts us to small ϵN_{Fr}^2.

If the remaining linear terms bring the present theory more in line with that of Whitaker (9), we could explain the scatter of experimental results. The theory shows that for given Reynolds and Weber numbers there is a range of values of the wave velocity that gives a steady wave train.

CONCLUDING REMARKS

We have derived an equation for the steady state of the free surface of a thin film of a liquid. We have shown that the equation has finite amplitude oscillations which are qualitatively in agreement with observed wave forms. The equation derived, however, is only valid provided that Reynolds number is less than about 20. When the Reynolds number is above 20, we would have to include more of the higher derivative terms to obtain more accurate results.

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NOTATION

a = nondimensional amplitude
A = nondimensional coefficient, $N_{fp} \sin\theta / (3N_{Fr}^4)^{1/3} a\epsilon^3$
B = nondimensional coefficient,
$$- a\epsilon^2 (3 + 27 N_{Fr}^2 \operatorname{cosec}\theta \, [(277/7! - 16\,\alpha/360) \cos\theta + N_{Fr}^2 (3879 - 7221\alpha + 3416\alpha^2)/8!]$$
c = wave velocity, cm./sec.
C = nondimensional coefficient,
$$a^2\epsilon (N_{Fr}^2 \operatorname{cosec}\theta \, (15\alpha/2 - 243/20) - 3\cot\theta)$$
D = nondimensional coefficient,
$$a\epsilon (N_{Fr}^2 \operatorname{cosec}\theta \, (15\alpha/8 - 81/40) - \cot\theta)$$
E = nondimensional coefficient, $(\alpha - 3)a$
F = nondimensional coefficient, $-3a^2$
g = acceleration due to gravity, cm./sec.^{2}
h = height above the bottom, cm.
h_0 = undisturbed height, cm.
n = normal to the free surface
$$[-(\partial h/\partial x), 1]/[1 + (\partial h/\partial x)^2]$$
N_{fp} = fluid properties number, dimensionless, $\Gamma/(g\nu^4)^{1/3}$
N_{Fr} = Froude number, dimensionless, $u/\sqrt{gh_0}$
N_{Re} = Reynolds number, dimensionless, Q/ν
N_{We} = Weber number, dimensionless
p = fluid pressure, dynes/sq.cm.
$\mathbf{p}$ = pressure tensor, dynes/sq.cm.
p_a = external (atmospheric) pressure, dynes/sq.cm.
Q = flow rate, sq.cm./sec., $u_0 h_0$
t = time, sec.
u, v = fluid velocities in the x and y directions, cm./sec.
u_0 = modulus of the average velocity over cross section, cm./sec., $(1/h_0) \int_0^{h_0} u \, dy$
x, y = Cartesian coordinates parallel to the wall and perpendicular to the wall, cm.
X = nondimensional χ coordinates $(\epsilon\chi/h_0)$
Y = nondimensional y coordinates (y/h_0)

Greek Letters

α = dimensionless wave velocity, c/u_0
Γ = kinematic surface tension, cc./sec.
ϵ = stretching parameter in the x direction (dimensionless)
η = scaled nondimensional disturbed height $(h - h_0)/a$
θ = angle of inclination of the wall, rad.
ν = kinematic viscosity, sq.cm./sec.
ρ = fluid density, g./cc.
χ = the variable $x + ct$, cm.

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